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(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2024

Mains

MATHEMATICS

Optional Paper-I

Solutions

1(a) Let H be a subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1,-2,5,-3)$, $v_2 = (2,3,1,-4)$, $v_3 = (3,8,-3,-5)$. Then find a basis and dimension of H, and extend the basis of H to a basis of $\mathbb{R}^4$

Sol Let $\mathbb{R}^4 = \{(a,b,c,d) / a,b,c,d \in \mathbb{R}\}$ be a V.S. over $\mathbb{R}$.

Let $S = \{v_1, v_2, v_3\} \subseteq \mathbb{R}^4$.

Given that H is a subspace of $\mathbb{R}^4$ spanned by S. then

$H = L(S) \subseteq \mathbb{R}^4$.

We have $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

clearly it is in echelon form and have two non-zero rows.

$\therefore \dim(H) = 2$.

and $S_1 = \{(1,-2,5,-3), (0,7,-9,2)\}$ is a basis of H.

Let us extend a basis of H to a basis of $\mathbb{R}^4$.

Let ~~us check~~ $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ be a basis of $\mathbb{R}^4$. (standard)

Let us check $(1,-2,5,-3)$, $(0,7,-9,2)$, $(0,0,1,0)$ and $(0,0,0,1)$ are L.I or not:

We have

$B = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ clearly it is in echelon form and has four non-zero rows.

$\therefore \dim(\mathbb{R}^4) = 4 =$ The number of L.I vectors.

$\therefore S_2 = \{(1,-2,5,-3),(0,7,-9,2)$
$(0,0,1,0),(0,0,0,1)\}$
is a basis of $\mathbb{R}^4$.

2.(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and $B = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$ over $\mathbb{R}$. Suppose that $T(v_1) = (1,1,0)$, $T(v_2) = (1,0,-1)$, $T(v_3) = (2,1,-1)$. Find a basis for the range space and null space of T.

Sol Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator.

Let $B = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$

Then (i) $L(B) = \mathbb{R}^3$

(ii) B is L.I.

Let $\alpha \in \mathbb{R}^3$ then $\alpha \in L(B)$

$\Rightarrow \alpha = x v_1 + y v_2 + z v_3$

Since $T: \mathbb{R}^3 \to \mathbb{R}$ is linear operator

$\therefore T(\alpha) = x T(v_1) + y T(v_2) + z T(v_3)$

$\Rightarrow T(\alpha) = x(1,1,0) + y(1,0,-1) + z(2,1,-1)$

$\Rightarrow T(\alpha) = (x+y+2z,\ x+z,\ -y-z)$.

Let $R(T) = \{T(\alpha) / \alpha \in \mathbb{R}^3\} \subseteq \mathbb{R}^3$

Then $R(T) = \{(x+y+2z,\ x+z,\ -y-z) / x,y,z \in \mathbb{R}\}$

Since $R(T)$ is spanned by $T(e_1), T(e_2)$, and $T(e_3)$.

we have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$\therefore$ It is in echelon form. and has two non-zero rows.

$\therefore \dim(R(T)) = 2$.

$\therefore$ The basis of $R(T)$ is $\{(1,1,0), (0,-1,-1)\}$.

We have

$N(T) = \{\alpha \in \mathbb{R}^3 / T(\alpha) = \hat{0}\} \subseteq \mathbb{R}^3$.

is a null space of $T: \mathbb{R}^3 \to \mathbb{R}$

Let $T(\alpha) = \bar{0}$

$\Rightarrow (x+y+2z,\ x+z,\ -y-z) = (0,0,0)$

$\Rightarrow x+y+2z=0$.

$\left.\begin{array}{l} x+z=0 \\ -y-z=0 \end{array}\right\} \Rightarrow x-y=0$.

$\Rightarrow \boxed{x=y}$

$\therefore y+y+2z=0$

$\Rightarrow \boxed{y=-z}$

$\therefore \boxed{x=-z}$

Since there is only one free variable say z.

$\therefore \dim(N(T)) = 1$.

and $N(T) = \{(-z, -z, z) / z \in \mathbb{R}\} \subseteq \mathbb{R}^3$

The basis of $N(T)$ is $\{(-1,-1,1)\}$

1. (c) Discuss the continuity of the function

$$f(x) = \begin{cases} \frac{1}{1-e^{-1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ for all values of x.

Sol^n Given function 'f' as

$$f(x) = \begin{cases} \frac{1}{1-e^{-1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ for all x.

$\because$ for $x \neq 0$ f is exponential function so it is continuous. Here we have to check continuity for $x=0$. $\because$ for a function to be continuous at $x=a$ it should have existing limit and equal to $f(a)$

i.e $\underset{x\to a^-}{Lt} f(x) = \underset{x\to a^+}{Lt} f(x) = f(a)$.

Here $x=0$: RHL: i.e $x \to 0^+$ $-\frac{1}{x} \to -\infty$

and $e^{-1/x} \to 0$ so $\underset{x\to 0^+}{Lt} f(x) = \frac{1}{1-0} = 1$

LHL: i.e $x \to 0^-$ $-\frac{1}{x} \to \infty$ and $e^{-\frac{1}{x}} \to \infty$

so $\underset{x\to 0^-}{Lt} f(x) = \frac{1}{1-\infty} = 0$

clearly LHL $\neq$ RHL i.e limit does not exist at $x=0$. so clearly LHL $\neq$ RHL $\neq f(0)$. Hence f is discontinuous at $x=0$ and the point of discontinuity is of the second kind.

1.(d) → Expand $\ln(x)$ in powers of $(x-1)$ by Taylor's theorem and hence find the value of $\ln(1.1)$ correct upto four decimal places.

Sol^n Taylor series is infinite series form of any function based on derivative of that function.

Taylor ~~Series~~ Theorem provides an approximation of a function near a given point using a polynomial derived from the function's derivatives.

i.e $f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \cdots + f^n(a)(x-a)^n + \cdots$

where $x \in$ nbd(a).

So using this $\ln(1+y)$ can be expanded where $y \in (-1, 1]$ as

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$

Let us take $y = x-1$ and put above

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

This is required expansion.

Now to find $\ln(1.1)$ i.e $x = 1.1$ So putting the values

$$\ln(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

upper term of higher power ignored as upto decimal 4 places required. So

$$\ln(1.1) = 0.1 - 0.005 + 0.0003 - 0.00002$$

$$= 0.0953 \quad \text{Ans.}$$

1.(e) Find the equation of the right circular cylinder which passes through the circle $x^2+y^2+z^2=9$, $x-y+z=3$

Sol^n: The direction ratios of the axis of the cylinder, which is perpendicular to the plane of the circle given by $x-y+z=3$ are $1,-1,1$.

So let one of the generators of the cylinder passing through any point (α,β,γ) on the cylinder be

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-1}=\frac{z-\gamma}{1}.$$

Any point on this generator at a distance r from (α,β,γ) is $(\alpha+r,\beta+r,\gamma+r)$.

If this point on the given circle, then we have

$$(\alpha+r)^2+(\beta-r)^2+(\gamma+r)^2=9,$$
$$(\alpha+r)-(\beta-r)+(\gamma+r)=3$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+2r(\alpha-\beta+\gamma)+3r^2=9,$$
$$\alpha-\beta+\gamma+3r=3$$

Eliminating r we get

$$\alpha^2+\beta^2+\gamma^2+2(\alpha-\beta+\gamma)\left(\frac{3-\alpha+\beta-\gamma}{3}\right)$$
$$+3\left[\frac{1}{3}(3-\alpha+\beta-\gamma)\right]^2=9$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+2(\alpha-\beta+\gamma)(3-\alpha+\beta-\gamma)+(3-\alpha+\beta-\gamma)^2=27$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+(3-\alpha+\beta-\gamma)(3+\alpha-\beta+\gamma)=27$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+9-(\alpha-\beta+\gamma)^2=27$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\alpha\beta-\alpha\gamma+\beta\gamma-9=0.$$

$\therefore$ The equation of the cylinder

i.e, the locus of $P(x_1, y_1, z_1)$ is

$$x^2+y^2+z^2+xy-xz+yz-9=0$$

2. (a) Consider a linear Transformation on R^3 over R defined by $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$. is T invertible? If yes, justify your answer and find T^{-1}.

Sol^n Given linear transform as $T: R^3 \to R^3$ when $R^3(R)$ is a vector space, such that $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$ for all $\alpha(x, y, z) \in R^3(R)$.

T can be invertible iff T is injective and surjective as well. or we can say if kernel of $T = \eta(T) = 0$ and $\rho(T) = \dim(R^3)$.

So, let us rewrite T as for any $\alpha(x,y,z)$

$T(x, y, z) = x(2,4,2) + y(0,-1,3) + z(0,0,-1)$

$\therefore$ $\rho(T)$ can be found from this definition as

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2+3R_3 \\ R_1 \to \frac{1}{2}R_1 \\ R_1 \to R_1 - 2R_2 \\ R_1 \to R_1 - R_3}]{\substack{R_3 \to -R_3 \\ R_2 \to -R_2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly the set $S = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (2,4,2)$ $\alpha_2(0,-1,3)$ $\alpha_3(0,0,-1)$ are L.I. vectors. And it's row reduced form is standard basis of $R^3(R)$ so clearly $L(S) = R^3$. Hence S is basis of range space of T. So $\dim(T) = \rho(T) = 3$. Now using rank-nullity theorem $\rho(T) + \eta(T) = \dim(R^3)$

$\Rightarrow 3 + \eta(T) = 3 \Rightarrow \eta(T) = 0$ or the range space for nullity of T is '0'

so Kernel of $T = \eta(T) = 0$ hence T is invertible.

Now To find T^{-1}: for any Linear transformation $T: U \to V$ and if T is invertible $\exists\ T^{-1}: V \to U$ such that for $\alpha \in U$, $\beta \in V$

$$T(\alpha) = \beta \Rightarrow T^{-1}(\beta) = \alpha.$$

let $\alpha \equiv (x, y, z)$ and $\beta \equiv (a, b, c)$ so

According to L.T. $T(\alpha) = \beta \Rightarrow (2x, 4x-y, 2x+3y-z) = a, b, c$

$\Rightarrow 2x = a \Rightarrow x = a/2$; $4x - y = b \Rightarrow 4\cdot\frac{a}{2} - y = b$
$\Rightarrow y = 2a - b$

and $2x + 3y - z = c \Rightarrow a + 3(2a-b) - z = c$
or $z = 7a - 3b - c$

Now $\because$ we know that $T^{-1}(\beta) = \alpha$ or
$T^{-1}(a, b, c) = (x, y, z)$

$$\Rightarrow T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$$

hence it is the required inverse of given Linear transformation T.

2.(b) If $u = \dfrac{(x+y)}{(1-xy)}$ and $v = \tan^{-1}x + \tan^{-1}y$, then find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If Yes, find the relationship.

Sol^n We are given functions U and V which are functions of $x, \& y$. as $u(x,y) = \dfrac{x+y}{1-xy}$

$V(x,y) = \tan^{-1}x + \tan^{-1}y$

To find $\dfrac{\partial(U,V)}{\partial(x,y)}$ which is $= J = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$

now $\dfrac{\partial U(x,y)}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{x+y}{1-xy}\right) = \dfrac{(1-xy)-(x+y)(-y)}{(1-xy)^2}$

$= \dfrac{1+y^2}{(1-xy)^2}$

$\dfrac{\partial U(x,y)}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{x+y}{1-xy}\right) = \dfrac{(1-xy)-(x+y)(-x)}{(1-xy)^2}$

$= \dfrac{1+x^2}{(1-xy)^2}$

$\dfrac{\partial V(x,y)}{\partial x} = \dfrac{\partial}{\partial x}\left(\tan^{-1}x + \tan^{-1}y\right) = \dfrac{1}{1+x^2}$

$\dfrac{\partial V(x,y)}{\partial y} = \dfrac{\partial}{\partial y}\left(\tan^{-1}x + \tan^{-1}y\right) = \dfrac{1}{1+y^2}$

now $\dfrac{\partial(U,V)}{\partial(x,y)} = |J| = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$

Now to find if U and V are functionally related or not we have check $|J| = 0$ or not

So $|J| = \frac{\cancel{1+y^2}}{(1-xy)^2}\cdot\frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2}\cdot\frac{1}{\cancel{1+x^2}}$

$= 0.$

So clearly $|J| = 0 \Rightarrow$ U and V are functionally dependent on each other or related.

and $\frac{\partial(U,V)}{\partial(x,y)} = 0.$

Now to find relation:

Let $\tan^{-1}x = w \Rightarrow x = \tan w$ and

$\tan^{-1}y = z \Rightarrow y = \tan z$

now putting these values $V = w + z$ —(i)

and $U = \frac{\tan w + \tan z}{1 - \tan w \tan z} = \tan(w+z).$

or $\boxed{U = \tan V}$ is the required relation.

2.(c) Find the image of the line $x=3-6t,\ y=2t,\ z=3+2t$ in the plane $3x+4y-5z+26=0$.

Sol^n: The given line is $x-3=-6t,\ y=2t,\ z-3=2t$

$$\Rightarrow \frac{x-3}{-6}=\frac{y-0}{2}=\frac{z-3}{2}=t \quad \text{— (1)}$$

Any point on this line is $A(3-6t, 2t, 3+2t)$ — (2)

If A lies on the plane $3x+4y-5z+26=0$, then we have

$$3(3-6t)+4(2t)-5(3+2t)+26=0$$
$$\Rightarrow t=-1$$

$\therefore$ from (2), the point A is $(9,-2,1)$, where A is the point of intersection of the given plane and the given line.

Also from (1) it is evident that any point on the given line is $C(3,0,3)$. Let B be the foot of the $\perp$lar from C on the given plane.

Now BC is a line $\perp$lar to the given plane i.e. it is normal to the given plane and as such the direction ratios of BC are $3,4,-5$ (the coefficients of x,y,z in the equation of the given plane).

$\therefore$ Equations of BC are $\frac{x-3}{3}=\frac{y-0}{4}=\frac{z-3}{-5}$

Any point on this line is $(3+3r, 4r, 3-5r)$. If this point is B i.e. if this point lies on the given plane then we have

$3(3+3r)+4(4r)-5(3-5r)+26=0$

$\Rightarrow r=-2/5$

$\therefore$ The point B is $\left(3-\frac{6}{5}, -\frac{8}{5}, 3+2\right)$

i.e. $\left(9/5, -8/5, 5\right)$

$\therefore$ The direction ratios of the projection AB of the given line are

$9-9/5, -2+8/5, 1-5$ i.e. $\frac{36}{5}, \frac{-2}{5}, -4$

i.e. $36, -2, -20$ i.e. $18, -1, -10$.

$\therefore$ The required equations of the projection AB are

$$\frac{x-9}{18}=\frac{y+2}{-1}=\frac{z-1}{-10}.$$

3(c) Let $V = M_{2\times 2}(\mathbb{R})$ denote a vectorspace over the field of real numbers. Find the Matrix of the linear mapping $\phi: V \to V$ given by $\phi(v) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} v$ w.r.t standard basis of $M_{2\times 2}(\mathbb{R})$ and hence find the rank of ϕ.
Is ϕ invertible? Justify your answer.

Sol Let $V = M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in \mathbb{R} \right\}$ be a v.s over $\mathbb{R}$.

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be a standard basis of V.

Given that $\phi: V \to V$ s.t
$\phi(v) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} v \quad \forall v \in V$.

Let $v = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in V$.

then $\phi: V \to V$ s.t $\phi(v) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$\Rightarrow \phi(v) = \begin{bmatrix} x+2z & y+2t \\ 3x-z & 3y-t \end{bmatrix} \quad \forall v \in V$.

Let $\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} -$

$\phi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} +$

$\phi\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}2&0\\-1&0\end{bmatrix}=2\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+(-1)\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}.$

$\phi\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&2\\0&-1\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+2\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}-1\begin{bmatrix}0&0\\0&1\end{bmatrix}.$

$\therefore$ The matrix of $\phi: V\to V$ w.r.t standard basis is.

$$\begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&-1&0\\0&3&0&-1\end{bmatrix}.$$

Let $R(\phi)=\left\{\begin{bmatrix}x+2z & y+2t\\3x-z & 3y-t\end{bmatrix} \Big/ x,y,z,t\in\mathbb{R}\right\}\subseteq V.$

let $A=\begin{bmatrix}x+2z & y+2t\\3x-z & 3y-t\end{bmatrix}\in R(\phi)$

then $A= x\begin{bmatrix}1&0\\3&0\end{bmatrix}+y\begin{bmatrix}0&1\\0&3\end{bmatrix}+z\begin{bmatrix}2&0\\-1&0\end{bmatrix}+t\begin{bmatrix}0&2\\0&-1\end{bmatrix}.$

$\in L(S)$

where $S=\left\{\begin{bmatrix}1&0\\3&0\end{bmatrix},\begin{bmatrix}0&1\\0&3\end{bmatrix},\begin{bmatrix}2&0\\-1&0\end{bmatrix},\begin{bmatrix}0&2\\0&-1\end{bmatrix}\right\}\subseteq R(\phi)$

$\therefore R(\phi)\subseteq L(S)$

$\because S\subseteq R(\phi)\Rightarrow L(S)\subseteq R(\phi)$

$$\phi\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}2&0\\-1&0\end{bmatrix}=2\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+-1\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

$$\phi\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&2\\0&-1\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+2\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}-1\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

$\therefore$ The matrix of $\phi: V \to V$ w.r.t standard basis is.

$$\begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&-1&0\\0&3&0&-1\end{bmatrix}.$$

Let $R(\phi)=\left\{\begin{bmatrix}x+2z & y+2t\\3x-z & 3y-t\end{bmatrix} \Big/ x,y,z,t \in \mathbb{R}\right\} \subseteq V.$

let $A=\begin{bmatrix}x+2z & y+2t\\3x-z & 3y-t\end{bmatrix} \in R(\phi)$

~~then~~ $A = x\begin{bmatrix}1&0\\3&0\end{bmatrix}+y\begin{bmatrix}0&1\\0&3\end{bmatrix}+z\begin{bmatrix}2&0\\-1&0\end{bmatrix}+t\begin{bmatrix}0&2\\0&-1\end{bmatrix}.$

$\in L(S)$

where $S=\left\{\begin{bmatrix}1&0\\3&0\end{bmatrix},\begin{bmatrix}0&1\\0&3\end{bmatrix},\begin{bmatrix}2&0\\-1&0\end{bmatrix},\begin{bmatrix}0&2\\0&-1\end{bmatrix}\right\} \subseteq R(\phi)$

$\therefore R(\phi) \subseteq L(S)$

$\because S \subseteq R(\phi) \Rightarrow L(S) \subseteq R(\phi)$

$\therefore N(\phi) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$\therefore \phi : V \to V$ is a non-singular L.T.

since $\dim(V) = 4$.

and $\dim(R(\phi)) = 2$.

$\therefore \dim(V) \neq \dim R(\phi)$.

$\therefore R(\phi) \neq V$

$\phi : V \to V$ is not onto.

$\therefore \phi^{-1} : V \to V$ does not exist.

$\therefore \phi$ is not invertible

3.(b). Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α.

Sol'n:

Let a cylinder of base radius 'r' and height h_1 is inscribed in a cone of height h and semivertical angle α.

In right-angled Δ^{le} AOB,

$$\tan\alpha = \frac{AB}{OA} = \frac{r}{h-h_1}$$

$$\Rightarrow r = (h-h_1)\tan\alpha.$$

$\therefore$ volume of the cylinder $V = \pi[(h-h_1)\tan\alpha]^2 h_1$.

$(\because V = \pi r^2 h_1)$

$$= \pi h_1 (h-h_1)^2 \tan^2\alpha \quad \text{—(1)}$$

Diff. (1) w.r.t. h_1, we get

$$\frac{dv}{dh_1} = \pi\tan^2\alpha\left[h_1 \cdot 2(h-h_1)(-1) + (h-h_1)^2\right].$$

$$= \pi\tan^2\alpha(h-h_1)(h-3h_1). \text{ —(2)}$$

For maximum volume V, $\frac{dv}{dh_1} = 0$

$$\Rightarrow \pi\tan^2\alpha(h-h_1)(h-3h_1) = 0$$

$$\Rightarrow h-h_1 = 0 \text{ (or) } h = 3h_1$$

$$\Rightarrow h = h_1 \text{ (or) } h_1 = \frac{h}{3}$$

$$\therefore h_1 = \frac{h}{3} \quad (\because h \neq h_1)$$

Again Diff (2) w.r.t. h_1,

$$\frac{d^2v}{dh_1^2} = \pi\tan^2\alpha\left[(h-h_1)(-3) + (h-3h_1)(-1)\right]$$

$$\left.\frac{d^2v}{dh_1^2}\right|_{h_1=\frac{h}{3}} = -2\pi h\tan^2\alpha < 0.$$

$\therefore$ volume is maximum for $h_1 = \frac{h}{3}$

$$\therefore V_{max} = \pi\tan^2\alpha\left(\frac{h}{3}\right)\cdot\left(h-\frac{h}{3}\right)^2 = \frac{4}{27}\pi h^3\tan^2\alpha.$$

3. (c) Find the vertex of the cone
$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0.$

Sol^n The given equation of cone when compared with the standard equation of 2-degree conic we get $a=4$, $b=-1$, $c=2$, $f=-\frac{3}{2}$, $g=0$, $h=1$, $u=6$, $v=-\frac{11}{2}$, $w=3$, $d=4$.

This general equation of conic will represent a cone if $D = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0.$

putting the values we get

$$\begin{vmatrix} 4 & 1 & 0 & 6 \\ 1 & -1 & -3/2 & -11/2 \\ 0 & -3/2 & 2 & 3 \\ 6 & -11/2 & 3 & 4 \end{vmatrix} = 4(-25) - 1(128) - 12(-38)$$
$= 0$. So $|D| = 0$

hence this 2nd degree general conic represents a cone with vertex at (x, y, z) say.

To find vertex: $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial t} = 0$ where $t = 1$. and

$f(x, y, z, t) = 4x^2 - y^2 + 2z^2 - 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2$

So $\frac{\partial f}{\partial x} = 8x + 2y + 12t = 0$ $\qquad \frac{\partial f}{\partial t} = 12x - 11y + 6z + 8t = 0$

$\frac{\partial f}{\partial z} = 4z - 3y + 6t = 0$

$\frac{\partial f}{\partial y} = -2y + 2x - 3z - 11t = 0$

Now solving the above 4 equations simultaneously and putting $t=1$ we get

$8x + 2y + 0z = -12$

~~$-2y+2z$~~

$2x - 2y - 3z = 11$

$0 - 3y + 4z = -6$

$12x - 11y + 6z = -8$

Solving we get $x=-1$, $y=-2$, $z=-3$

So vertex is (x, y, z) i.e

Vertex of given cone is $(-1, -2, -3)$ Ans.

4. (a) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ be a 3×3 matrix. Find the eigenvalues and the corresponding eigenvectors of A. Hence find the eigenvalues and the corresponding eigenvectors of A^{-15}, where $A^{-15} = (A^{-1})^{15}$.

Sol^n For, the given A, $\det(A) = 8 \neq 0$ so clearly A^{-1} exist.

Now if e is eigenvector of A then for some 'λ' $Ae = \lambda e \Rightarrow (A-\lambda I)e = 0$ is

$|A - \lambda I| = 0$ (as $e \neq 0$) is called characteristic equation.

here $\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$ it can be simplified as

$$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{trace of adj}A)\lambda - |A| = 0$$

here $\text{adj } A = (A^{-1})\cdot|A| = \begin{pmatrix} -4 & 2 & 4 \\ 2 & -7 & 2 \\ 4 & 2 & -4 \end{pmatrix}$

trace of adj $A = -15$

trace of $A = 6$

So characteristic equation is

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$ solving for 'λ'

we get $(\lambda-8)(\lambda+1)^2 = 0 \Rightarrow \lambda = 8, -1, -1$.

For eigen vector for $\lambda = 8$:
$$-5x + 2y + 4z = 0$$
$$2x - 8y + 2z = 0$$
$$4x + 2y - 5z = 0$$

Solving for (x, y, z) we get

$(t, t/2, t) = t(1, 1/2, 1)$ hence

eigen vector is $(1, 1/2, 1)$ for $\lambda = 8$.

Eigen vector $\lambda = -1$

$4x + 2y + 4z = 0$
$2x + y + 2z = 0$
$4x + 2y + 8z = 0$

so solving we get

$(x, y, z) \equiv (-t/_2, t, 0) = t(-1/_2, 1, 0)$

hence eigenvector for $\lambda = -1$ are $(-1/_2, 1, 0)$.

and $(-1, 0, 1)$.

as $(x, y, z) = \begin{pmatrix} -\frac{1}{2}y - z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$\because$ we know that if λ is eigenvalue of A then eigenvalue of A^{-1} will be $1/\lambda$.

So eigenvalues of $A^{-1} \equiv 1/8, \frac{1}{-1}, \frac{1}{-1} \equiv (\frac{1}{8}, -1, -1)$

also if λ is eigenvalue of A then eigenvalue of A^n will be λ^n. similarly for $(A^{-1})^{15}$ we get eigenvalues as $(\frac{1}{8^{15}}, -1, -1)$.

Now Eigenvectors corresponding to $(A^{-1})^{15}$ for $\lambda = -1$ will be same as $(-1/_2, 1, 0)$ & $(-1, 0, 1)$.

And for $\lambda = 1/8^{15}$ for $(A^{-1})^{15}$ will also be same as $(1, 1/_2, 1)$.

4(b) → Using double integration, find the area lying inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$.

Sol^n: Eliminating r between the given equations of the cardioid $r=a(1+\cos\theta)$ and the circle $r=a$ we have

$$a=a(1+\cos\theta) \text{ (or) } \cos\theta=0 \text{ i.e. } \theta=\pm\pi/2$$

Thus the region of integration A is enclosed by

$$r=a,\ r=a(1+\cos\theta),\ \theta=-\pi/2,\ \theta=\pi/2$$

$\therefore$ The required area $=\iint_A r\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,d\theta\,dr$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[a^2(1+\cos\theta)^2-a^2\right]d\theta$$

$$=\frac{a^2}{2}\int_{-\pi/2}^{\pi/2}(1+\cos^2\theta+2\cos\theta-1)\,d\theta$$

$$=\frac{a^2}{2}\cdot 2\int_0^{\pi/2}\left[\cos^2\theta+2\cos\theta\right]d\theta$$

$$=a^2\left[\frac{1}{2}\cdot\frac{1}{2}\pi+2\{\sin\theta\}_0^{\pi/2}\right]$$

$$=a^2\left[\pi/4+2\right]$$

$$=\frac{a^2}{4}(\pi+8)$$

4.(c) Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere
$x^2+y^2+z^2-4x+6y+4=0$

Sol'n: The given plane $3x+2y-z+2=0$ is a tangent plane to the required sphere at $A(1,-2,1)$,
So the line joining the centre C of the sphere & the point $A(1,-2,1)$ must be at right angles to this plane.

Hence the equation of the line AC is

$$\frac{1}{3}(x-1)=\frac{1}{2}(y+2)=-(z-1) \quad \text{—— (1)}$$

the direction cosines of the line AC are the coefficients of x,y,z in the given plane.

Any point on this line (1) is $(1+3r,-2+2r,1-r)$ and can be taken as the centre C of the sphere. Also the radius of the sphere is CA

i.e. $\sqrt{[(1+3r-1)^2+(-2+2r+2)^2+(1-r-1)^2]}$ i.e. $r\sqrt{14}$.

Now the centre and radius of the given sphere are $C'(2,-3,0)$ and $\sqrt{2^2+3^2-4}=3$.

If the two spheres cut orthogonally, then we have

$$(CP)^2+(C'P)^2=(CC')^2$$

$\Rightarrow (r\sqrt{14})^2+(3)^2=(1+3r-2)^2+(-2+2r+3)^2+(1-r-0)^2$

$\Rightarrow 14r^2+9=(3r-1)^2+(2r-1)^2+(1-r)^2$

$\Rightarrow r=-3/2$

$\therefore$ The centre C of the sphere is $(1+3r,-2+2r,1-r)$, where $r=-3/2$

i.e. $[1-9/2,-2-3,1+(3/2)] \Rightarrow (-7/2,-5,5/2)$

and radius is $r\sqrt{14} \Rightarrow \frac{3}{2}\sqrt{14}$,

$\therefore$ The equation of the sphere is

$$\{x+(7/2)\}^2+(y+5)^2+\{z-(5/2)\}^2=\left[-\tfrac{3}{2}\sqrt{14}\right]^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12=0.$$

5(a) Find the orthogonal trajectories of the family of curves $r = c(\sec\theta + \tan\theta)$, where c is a parameter.

Sol^n:

Given, the family of curves

$$r = c(\sec\theta + \tan\theta) \quad \text{--- (1)}$$

where c is the parameter.

Taking logarithm of (1) both sides, we get

$$\log r = \log c + \log(\sec\theta + \tan\theta) \quad \text{--- (2)}$$

Differentiating eqn (2) both sides, w.r.t θ, we get

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = \frac{(\sec\theta\tan\theta + \sec^2\theta)}{\sec\theta + \tan\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \sec\theta \quad \text{--- (3)}$$

which is the differential equation of the family of curves (1).

Now, replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in eqn (3), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \sec\theta$$

$$-r\frac{d\theta}{dr} = \sec\theta$$

$$\Rightarrow \frac{1}{\sec\theta}d\theta = -\frac{dr}{r} \Rightarrow \cos\theta\, d\theta = -\frac{dr}{r}$$

On integration, we get

$$\sin\theta = -\log r + \log b \quad , \text{ b is an arbitrary constant.}$$

$$\sin\theta = \log\frac{b}{r}$$

$$\frac{b}{r} = e^{\sin\theta} \Rightarrow b = re^{\sin\theta}.$$

which is the required orthogonal trajectory.

5(b) Solve the integral equation

$y(t) = \cos t + \int_0^t y(x) \cos(t-x)\, dx$

using Laplace transform.

Solⁿ: Given that

$$y(t) = \cos t + \int_0^t y(x) \cos(t-x) \quad \text{—①}$$

Using the definition of convolution, eqn ① can be re-written as

$$y(t) = \cos t + y(t) * \cos t$$

Let $L\{y(t)\} = f(p)$.

Applying the Laplace transform of ①, we have

$$L\{y(t)\} = L\{\cos t\} + L\{y(t) * \cos t\}$$

$$L\{y(t)\} = \frac{p}{p^2+1} + L\{y(t)\}\, L\{\cos t\}$$

$$L\{y(t)\} = \frac{p}{p^2+1} + L\{y(t)\} \frac{p}{p^2+1}$$

$$f(p) = \frac{p}{p^2+1} + f(p) \frac{p}{p^2+1}$$

$$f(p)\left\{1 - \frac{p}{p^2+1}\right\} = \frac{p}{p^2+1}$$

$$\left(\frac{p^2+1-p}{p^2+1}\right) f(p) = \frac{p}{p^2+1}$$

$$f(p) = \frac{p}{p^2-p+1}$$

Applying inverse Laplace transform we get

$$L^{-1}\{y(p)\} = L^{-1}\left\{\frac{p}{p^2-p+1}\right\}$$

$$y(t) = L^{-1}\left\{\frac{p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= e^{\frac{t}{2}} L^{-1}\left\{\frac{p}{p^2+\frac{3}{4}}\right\}$$

$$= e^{t/2} L^{-1}\left\{\frac{p}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$y(t) = e^{t/2}\cos\frac{\sqrt{3}}{2}t$$

5.(c)(ii) At any time 't' in (seconds), the coterminous edges of a variable parallelopiped are represented by the vectors $\vec{\alpha} = t\hat{i} + (t+1)\hat{j} + (2t+1)\hat{k}$

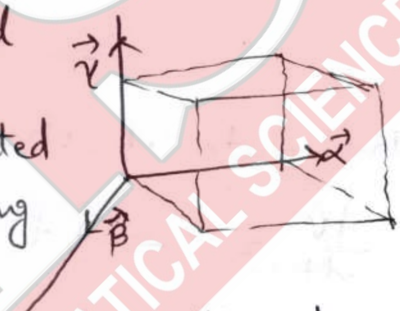
$$\vec{\beta} = 2t\hat{i} + (3t-1)\hat{j} + t\hat{k}$$

$$\vec{\gamma} = \hat{i} + 3t\hat{j} + \hat{k}$$

What is the rate of change of the vectorial area of the parallelogram, whose coterminous edges are $\vec{\alpha}$ & $\vec{\gamma}$? Also find the rate of change of the volume of the parallelopiped at $t=1$ second.

Sol^n

Let the given parallelopiped be as shown in figure.



The sides of are represented as vectors which is changing with time.

Now, Area of any parallelogram with edges as $\vec{\alpha}, \vec{\gamma}$ is $\vec{A} = \vec{\alpha} \times \vec{\gamma}$

So $\vec{A} = \begin{vmatrix} i & j & k \\ t & t+1 & 2t+1 \\ 1 & 3t & 1 \end{vmatrix} = \hat{i}(t+1-6t^2-3t) - \hat{j}(t-2t-1) + \hat{k}(3t^2-t-1)$

$$= (-6t^2-2t+1)\hat{i} + (t+1)\hat{j} + (3t^2-t-1)\hat{k}$$

Now rate of change of vectorial area is

$$\frac{d\vec{A}}{dt} = (-12t-2)\hat{i} + \hat{j} + (6t-1)\hat{k}$$

Now Volume of parallelopiped with sides $\vec{\alpha}, \vec{\gamma}, \vec{\beta}$ is given as $|\vec{\alpha} \times \vec{\gamma} \cdot \vec{\beta}|$

So Volume $= \vec{A} \cdot \vec{B}$

$= (-6t^2 - 2t + 1) \cdot 2t + (3t-1) \cdot (t+1) + t \cdot (3t^2 - t - 1)$

$= -12t^3 - 4t^2 + 2t + 3t^2 - t + 3t - 1 + 3t^3 - t^2 - t$

$= -9t^3 - 2t^2 + 3t - 1$

So volume at $t = 1$ second

$V_{at\ t=1} = |-9 - 2 + 3 - 1|$

$= 9 \text{ units}^3.$

Now rate of change of volume is

$\frac{dV}{dt} = -27t^2 - 4t + 3$

$\left.\frac{dV}{dt}\right|_{t=1} = (-27 - 4 + 3)$

$= -28$ Ans

5(d) A solid hemisphere rests in equilibrium on a solid sphere of equal radius. Determine the stability of the equilibrium in the two situations — (i) when the curved surface and (ii) when the flat surface of the hemisphere rests on the sphere.

Sol^n: (i) when the curved surface of the hemisphere rests on the sphere.

A hemisphere of centre O' rests on a sphere of centre O with its curved surface in contact with the sphere. The point of contact is A and $OA = O'A = a$ (say).

Also the line OAO' is vertical.

If G is the centre of gravity of the hemisphere, then G lies on $O'A$ and $O'G = 3/8\,a$.

Here ρ_1 = the radius of curvature of the upper body at the point of contact = the radius of the hemisphere $= a$, and ρ_2 = the radius of curvature of the lower body at the point of contact A.

Also h = the height of the centre of gravity of the upper body above the point of contact A.

$$= AG = O'A - O'G = a - \tfrac{3}{8}a = \tfrac{5}{8}a$$

we have $\dfrac{1}{h} = \dfrac{1}{5a/8} = 8/5a$

& $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} = \dfrac{1}{a} + \dfrac{1}{a} = \dfrac{2}{a} = \dfrac{10}{5a}$

Thus $\dfrac{1}{h} < \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$. Hence the equilibrium is unstable in this case.

(ii) when the flat surface of the hemisphere rests on the sphere.

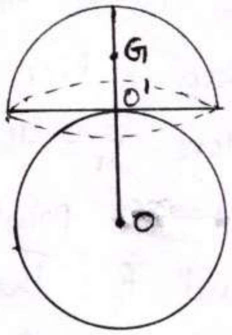
In this case a hemisphere of centre O' rests on a sphere of centre O and equal radius a with its flat surface (i.e. the plane base) in contact with the sphere. The point of contact is O' and G is the C.G. of the hemisphere.

Here ρ_1 = the radius of curvature of the upper body at the point of contact = ∞

[Note that the base of the hemisphere touches the sphere along a straight line].

and ρ_2 = the radius of curvature of the lower body at the point of contact = the radius of the sphere = a

Also h = the height of the C.G of the hemisphere above the point of contact O' = $O'G = \frac{3}{8}a$



we have $\frac{1}{h} = \frac{1}{3a/8} = \frac{8}{3a}$

and $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\infty} + \frac{1}{a} = 0 + \frac{1}{a} = \frac{1}{a} = \frac{3}{3a}$

Obviously $\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$. Hence in this case the equilibrium is stable.

5. (e) (i) Let C be a plane curve $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j}$, where f and g have second-order derivatives. Show that the curvature at a point is given by

$$K = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{\left([f'(t)]^2 + [g'(t)]^2\right)^{3/2}}.$$

What is the value of torsion τ at any point of this curve?

Sol^n Given the curve C as $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j}$ where f'' & g'' exists.

We know that $K = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$ where $\dot{\vec{r}} = \dfrac{d\vec{r}}{dt}$, $\ddot{\vec{r}} = \dfrac{d^2\vec{r}}{dt^2}$

$\because \dfrac{d\vec{r}}{ds} = \dfrac{d\vec{r}}{dt}\dfrac{dt}{ds}$ using this K in $\vec{r}$ calculated.

$\dfrac{d\vec{r}}{dt} = f'(t)\hat{i} + g'(t)\hat{j}$ so $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$

$\dfrac{d^2\vec{r}}{dt^2} = f''(t)\hat{i} + g''(t)\hat{j}$ Now putting these values for K we got

$$K = \frac{|(f'(t)\hat{i} + g'(t)\hat{j}) \times (f''(t)\hat{i} + g''(t)\hat{j})|}{\left(\sqrt{[f'(t)]^2 + [g'(t)]^2}\right)^3}$$

$$= \frac{|f'(t)g''(t)\hat{k} - g'(t)f''(t)\hat{k}|}{\left([f'(t)]^2 + [g'(t)]^2\right)^{3/2}}$$

$$= \frac{|f'(t)g''(t) - g'(t)f''(t)|}{\left([f'(t)]^2 + [g'(t)]^2\right)^{3/2}}$$ hence proved.

Now $\tau = [\dot{\vec{r}} \times \ddot{\vec{r}} \cdot \dddot{\vec{r}}] \Big/ |\dot{\vec{r}} \times \ddot{\vec{r}}|^2$

So $\dddot{\vec{r}} = \frac{d}{dt}\left(\frac{d^2\vec{r}}{dt^2}\right) = 0$ as f and g are having second order derivatives.

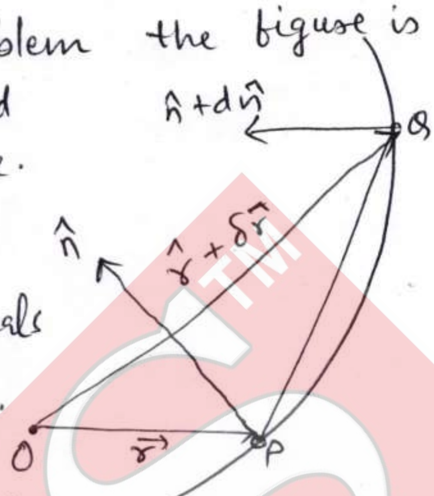
So $\tau = 0$ for any point on curve.

5. (e) (ii) Prove that the principal normals at consecutive points do not intersect unless $\tau = 0$.

Sol^n According to given problem the figure is.

let us take $P(\vec{r})$ and $Q(\vec{r}+\delta\vec{r})$ on the curve.

Also let $(\hat{n})$ and $(\hat{n}+d\hat{n})$ be unit principal normals at P and Q respectively.



As given principal normals at P and Q intersect, in this case $\hat{n}$, $\hat{n}+d\hat{n}$ and $d\hat{r}$ must be coplanar. This means $[\hat{n}\ \ \hat{n}+d\hat{n}\ \ d\hat{r}] = 0$ [Condition for coplanarity].

$\Rightarrow [\hat{n}\ \ \hat{n}\ \ d\hat{r}] = 0 \quad (\because \hat{n}\times\hat{n} = 0)$

$\Rightarrow \left[\hat{n}\ \ \frac{d\hat{n}}{ds},\ \frac{d\hat{r}}{ds}\right] = 0 \qquad \because \frac{d\hat{n}}{ds} = \tau\hat{b} - ki$

$\Rightarrow [\hat{n},\ \tau\hat{b} - ki,\ i] = 0 \Rightarrow [\hat{n}, \tau\hat{b}, i] - k[\hat{n}, i, i] = 0$

$\because [\hat{n}\ i\ i] = 0$ we left with

$\tau[\hat{n}\ \hat{b}\ i] = 0$ clearly $[\hat{n}\ \hat{b}\ i] \neq 0$

so $\tau = 0$.

Hence proved that principal normals to intersect ~~at~~ ~~con~~ $\tau = 0$.

6(b) A particle executes Simple haromic motion such that in two of its positions, velocities are u and v, and the two corresponding accelerations are f_1 and f_2. For what value(s) of k, the distance between the two positions is $k(v^2-u^2)$? show also that the amplitude of the motion is

$$\frac{1}{f_2^2-f_1^2}\left[(u^2-v^2)(u^2f_2^2-v^2f_1^2)\right]^{1/2}.$$

Sol^n: Let the equation of the S.H.M with Centre as origin be $d^2x/dt^2=-\mu x$

If a be the amplitude of the motion, we have

$$\left(\frac{dx}{dt}\right)^2=\mu(a^2-x^2)$$

where dx/dt is the velocity at a distance x from the centre.

Let x_1 and x_2 be the distances from the Centre of the two positions where u and v are the velocities and f_1 & f_2 are the accelerations respectively. Then

$$f_1=\mu x_1 \quad \text{—①} \qquad f_2=\mu x_2 \quad \text{—②}$$

$$u^2=\mu(a^2-x_1^2) \quad \text{—③}$$

and $$v^2=\mu(a^2-x_2^2) \quad \text{—④}$$

Adding ① & ②, we get $f_1+f_2=\mu(x_1+x_2)$ —⑤

Also substracting ③ from ④, we get

$$v^2-u^2=\mu(x_1^2-x_2^2)=\mu(x_1-x_2)(x_1+x_2)$$
$$=(f_1+f_2)(x_1-x_2) \quad \text{from ⑤}$$

$\therefore (x_1 - x_2) = (v^2 - u^2)/(f_1 + f_2)$.

$\Rightarrow (x_1 - x_2) = k(v^2 - u^2) \qquad \left[\because k = \frac{1}{f_1 + f_2}\right]$

This gives the distance between the two positions.

Now to get the amplitude a it is obvious that we have to eliminate x_1, x_2 and μ from equations ①, ②, ③ & ④. Substituting for x_1 and x_2 from ① & ② in ③ & ④, we have

$$u^2 = \mu\left(a^2 - \frac{f_1^2}{\mu^2}\right) \quad \text{i.e.} \quad a^2\mu^2 - u^2\mu - f_1^2 = 0 \quad \text{—⑥}$$

and $$v^2 = \mu\left(a^2 - \frac{f_2^2}{\mu^2}\right) \quad \text{i.e.} \quad a^2\mu^2 - v^2\mu - f_2^2 = 0 \quad \text{—⑦}$$

By the method of cross multiplication, we have from ⑥ and ⑦

$$\frac{\mu^2}{u^2f_2^2 - v^2f_1^2} = \frac{\mu}{-a^2f_1^2 + a^2f_2^2} = \frac{1}{a^2u^2 - a^2v^2}$$

Equating the two values of μ^2 found from the above equations, we get

$$\frac{f_1^2v^2 - u^2f_2^2}{a^2(v^2 - u^2)} = \left[\frac{a^2(f_1^2 - f_2^2)}{a^2(v^2 - u^2)}\right]^2$$

$$\Rightarrow \frac{f_1^2v^2 - u^2f_2^2}{a^2(v^2 - u^2)} = \frac{(f_1^2 - f_2^2)^2}{(v^2 - u^2)^2}$$

$$\therefore a^2 = \frac{(f_1^2v^2 - f_2^2u^2)(v^2 - u^2)}{(f_1^2 - f_2^2)^2}$$

$$\therefore a = \frac{[(v^2 - u^2)(f_1^2v^2 - f_2^2u^2)]^{1/2}}{f_1^2 - f_2^2}$$

6(c)(i) Find the second solution of the differential equation $xy''+(x-1)y'-y=0$ using $u(x)=-e^{-x}$ as one of the solutions.

Sol^n: Given differential equation is

$$xy''+(x-1)y'-y=0 \quad \text{——} (1)$$

comparing eqn (1) with

$$p(x)y''+q(x)\,y'+r(x)y=0$$

we have $p(x)=x$, $q(x)=(x-1)$, $r(x)=-1$

Also here $f(x)=-e^{-x}$.

Hence the second solution $= v(-e^{-x})$

where $$v=\int \frac{e^{-\int \frac{q(x)}{p(x)}dx}}{\{f(x)\}^2}\,dx \quad \text{——} (2).$$

Now $$\int \frac{q(x)}{p(x)}dx=\int\frac{(x-1)}{x}dx=\int\left(1-\frac{1}{x}\right)dx$$
$$=x-\log x$$

$$e^{-\int \frac{q(x)}{p(x)}dx}=e^{-(x-\log x)}$$
$$=e^{-x+\log x}=e^{-x}x$$

$\therefore$ from (2),

$$v=\int\frac{xe^{-x}}{(-e^{-x})^2}\,dx=\int\frac{xe^{-x}}{e^{-2x}}\,dx$$

$= \int x e^{x}\, dx$

$= x e^{x} - \int 1 \cdot e^{x}\, dx$

$= x e^{x} - e^{x}$

$v = (x-1) e^{x}$

$\therefore$ Required second solution

$= v(-e^{-x})$

$= (x-1)e^{x}(-e^{-x})$

$= -(x-1)$

$= 1-x$

6.(c)(ii) → Find the general solution of the differential equation $x^2y''-2xy'+2y=x^3\sin x$ by the method of variation of parameters.

Solution: Given

$$x^2y''-2xy'+2y=x^3\sin x \quad \text{——} (1)$$

Divide equation (1) by x^2.

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=x\sin x$$

$[\because R=x\sin x]$

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=0 \quad \text{homogeneous equation}$$

$$x^2y''-2xy'+2y=0$$

$$(x^2y''-2xy'+2)y=0$$

$$(x^2D^2-2xD+2)y=0 \qquad [\because D\equiv\frac{d}{dx}]$$

put $x=e^z$ and $D_1\equiv\frac{d}{dz}$

Then, we get

$$(D_1(D_1-1)-2(D_1)+2)y=0$$

$$\Rightarrow (D_1^2-3D_1+2)y=0$$

$$(D_1-2)(D_1-1)=0$$

$$D=2,1.$$

$$y_c=c_1e^z+c_2e^{2z}$$

In terms of x.

$$y_c = c_1 x + c_2 x^2 \quad [\because x = e^z]$$

where $u = x \qquad v = x^2$

$u' = 1 \qquad v' = 2x.$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

for $P.I = A.u + Bv.$

where $A = -\int \frac{vR}{W} dx$ and $B = \int \frac{uR}{W} dx.$

$$A = -\int \frac{\cancel{x^2} . x \sin x}{\cancel{x^2}} dx$$

$$A = -(-x\cos x + \sin x) = x\cos x - \sin x + c_1$$

$$B = \int \frac{x . \cancel{x} \sin x}{\cancel{x^2}} dx = \int \sin x \, dx$$

$$B = -\cos x + c_2$$

$$P.I = x . [x\cos x - \sin x] + x^2 [-\cos x]$$

$$P.I. = x^2 \cancel{\cos x} - x\sin x + (-x^2 \cancel{\cos x})$$

$$\boxed{P.I = -x \sin x}$$

$$z = y_c + y_p$$

$$\boxed{z = c_1 x + c_2 x^2 - x\sin x}$$

Required general solution.

7(a) State uniqueness theorem for the existence of unique solution of the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ in the rectangular region $R: |x-x_0| \le a,\ |y-y_0| \le b$.

Test the existence and uniqueness of the solution of the initial value problem $\frac{dy}{dx} = 2\sqrt{y}$, $y(1) = 0$, in a suitable rectangle R. If more than one solution exist, then find ~~the~~ all the solutions.

Sol'n:

Statement:

Let $f(x,y)$ be continuous in a domain D of the (x,y) plane and let M be a constant such that $|f(x,y)| \le M$ in D.

Let $f(x,y)$ satisfy in D the Lipschitz condition in y namely

$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|,$$

where the constant K is independent of x, y_1, y_2

Let the rectangle R, defined by $|x-x_0| \le a$, $|y-y_0| \le b$, lie in D where $Ma < b$. Then for $|x-x_0| \le a$, the differential equation $\frac{dy}{dx} = f(x,y)$ has a unique solution $y = y(x)$ for which $y(x_0) = y_0$.

7(b) A heavy particle hanging vertically from a fixed point by a light inextensible string of length l starts to move with initial velocity u in a circle so as to make a complete revolution in a vertical plane. Show that the sum of tensions at the ends of any diameter is constant.

Sol'n: The tension T in the string in any position given by

$$T = \frac{m}{l}(u^2 - 2lg + 3lg\cos\theta) \quad \text{--- (1)}$$

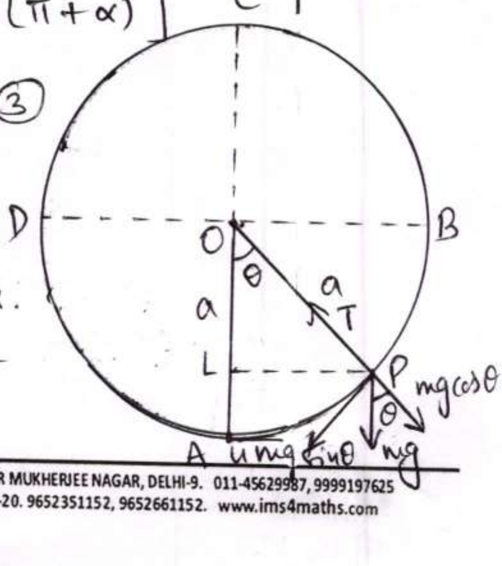
where θ is the angle which the string makes with OA.

Now take any diameter of the circle. If at one end of this diameter we have $\theta = \alpha$, then at the other end we shall have $\theta = \pi + \alpha$. Let T_1 and T_2 be the tensions at these ends i.e. $T = T_1$ when $\theta = \alpha$ and $T = T_2$ when $\theta = \pi + \alpha$.

Then from (1), we have

$$T_1 = \frac{m}{l}(u^2 - 2lg + 3lg\cos\alpha) \quad \text{--- (2)}$$

and $T_2 = \frac{m}{l}[u^2 - 2lg + 3lg\cos(\pi + \alpha)]$



$$\Rightarrow T_2 = \frac{m}{l}(u^2 - 2lg - 3lg\cos\alpha) \quad \text{--- (3)}$$

Adding (2) & (3), we have

$$T_1 + T_2 = 2\frac{m}{l}(u^2 - 2lg)$$

which is constant, as it is independent of α.

Hence the sum of the tensions at the ends of any diameter is constant.

7(c) verify the Stoke's theorem for the vector field $\bar{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ on the surface S which is the part of the cylinder $z = 1 - x^2$ for $0 \le x \le 1$, $-2 \le y \le 2$; S is oriented upwards.

Sol^n: By stoke's theorem, we have

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n}\, ds \quad \text{--- (1)}$$

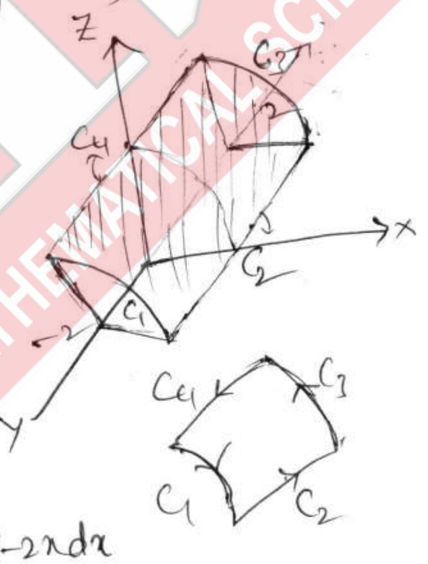
Let us verify (1).

Given that $F = xyi + yzj + xzk$.

we have

$$\oint_C F \cdot dr = \oint xy\,dx + yz\,dy + xz\,dz$$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$



Along C_1:

$x^2 + z = 1$

$y = -2, \Rightarrow dy = 0$

$$\therefore \int_{C_1} F \cdot dr = \int_0^1 -2x\,dx + x(1-x^2)(-2x\,dx)$$

$$= \int_0^1 (-2x - 2x^2 + 2x^4)\,dx$$

$$= \left[-x^2 - \frac{2x^3}{3} + \frac{2x^5}{5}\right]_0^1 = -1 - \frac{2}{3} + \frac{2}{5} = \frac{-19}{15}$$

Along C_2: $z = 0$, $x = 1 \Rightarrow dx = 0$, $dz = 0$

$$\int_{C_2} F \cdot dr = \int 0 = 0.$$

Along C_3: $y=2 \Rightarrow dy=0$

$$\int_{C_3} F\cdot dr = \int_1^0 2x\,dx + x(1-x^2)(-2x\,dx)$$

$$= \int_1^0 (2x - 2x^2 + 2x^4)\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3 + \frac{2}{5}x^5\right]_1^0 = 0 - \left(1 - \frac{2}{3} + \frac{2}{5}\right)$$

$$= \frac{-11}{15}$$

Along C_4: $z=1$, $x=0$
$\Rightarrow dz=0$, $dx=0$

$$\int_{C_4} F\cdot dr = \int_2^{-2} y(1)\,dy = \frac{y^2}{2} = 0$$

$$\therefore \oint_C F\cdot dr = \frac{-19}{15} - \frac{11}{15} = \frac{-30}{15} = -2$$

Now let us find $\iint_S \text{curl} F\cdot \hat{n}\,ds$

we have.

$$\text{curl} f = \nabla\times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$$

$$= -(y\hat{i} + z\hat{j} + x\hat{k})$$

If n is a unit vector along outward drawn normal at any point (x,y,z) on the surface S. i.e, the surface $\phi(x,y,z) = x^2+z = 1$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+\hat{k}}{\sqrt{4x^2+1}} \quad \text{and } ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\frac{1}{\sqrt{4x^2+1}}}$$

$$\therefore \iint_S (\nabla\times F)\cdot\hat{n}\,ds = \iint_S -(y\hat{i}+z\hat{j}+x\hat{k})\cdot\frac{2x\hat{i}+\hat{k}}{\sqrt{4x^2+1}}\cdot\frac{dx\,dy}{\sqrt{4x^2+1}}$$

$$= -\int_{x=0}^{1}\int_{y=-2}^{2} (2xy+x)\,dx\,dy = -\int (y^2+y)x\,dx$$

$$= -\int_{-1}^{1} 4x\,dx = -2.$$

Hence Stoke's theorem verified.

8(a) Using Laplace transform, solve the initial value problem,

$y''+2y'+5y=\delta(t-2)$, $y(0)=0$, $y'(0)=0$

where $\delta(t-2)$ denotes the Dirac delta function.

Sol^n:

Given that

$y''+2y'+5y=\delta(t-2)$

Taking Laplace transform on both sides.

we get

$L\{y''\}+2L\{y'\}+5L\{y\}=L\{\delta(t-2)\}$

$p^2L\{y\}-py(0)-y'(0)+2\left[pL\{y\}-y(0)\right]+5L\{y\}=L\{\delta(t-2)\}$

$p^2L\{y\}+2pL\{y\}+5L\{y\}=e^{-2p}$

$\boxed{\begin{aligned}&\because \text{Dirac Delta function:}\\ &L\{\delta(t-a)\}=\int_0^\infty e^{-pt}\delta(t-a)\,dt\\ &\qquad=e^{-ap} \quad \text{provided } a>0\end{aligned}}$

$\left[p^2+2p+5\right]L\{y\}=e^{-2p}$

$L\{y\}=\dfrac{e^{-2p}}{p^2+2p+5}$

$$L\{y\} = \frac{e^{-2p}}{(p+1)^2+4}$$

Taking inverse Laplace transform

$$y = e^{-2p}\, L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$$

$$= e^{-2p} e^{-t} L^{-1}\left\{\frac{1}{p^2+4}\right\}$$

$$= e^{-t} e^{-2p} \left(\frac{1}{2}\right) \sin 2t.$$

Ex 1: $L\{\delta(t-a)\} = e^{-ap}$

2: $L\{f(t)\,\delta(t-a)\} = f(a)\, e^{-ap}$

8 (b) Using Gauss Divergence theorem, evaluate the integral $\iint_S (y^2\hat{i} + xz^3\hat{j} + (z-1)^2\hat{k})\cdot\hat{n}\, dS$ over the region bounded by the cylinder $x^2+y^2=16$ and the planes $z=1$ and $z=5$.

Soln

Given $\vec{F} = y^2\hat{i} + xz^3\hat{j} + (z-1)^2\hat{k}$

Gauss Divergence theorem states that surface integral of $\vec{F}$ over $\vec{S}$ is equal to volume integral of $\vec{F}$ over V enclosed by $\vec{S}$.

Here Given surface is cylinder as

$x^2+y^2=16$

so by GDT

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV \qquad z=z.$$

Here, $\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{F}$

$$= 0 + 0 + 2(z-1) = 2z-2$$

Let us convert in polar coordinates for cylinder

$x = r\cos\theta \quad y = r\sin\theta \quad z = z. \qquad dv = dx\,dy\,dz$

$= r\,dr\,d\theta\,dz$

here limits are $r\in[0,4]$

$\theta\in[0,2\pi] \qquad z\in[1,5]$

$$\text{So, } I = \iiint_V \nabla\cdot\vec{F}\, dv = \iiint_V (2z-2)\,dx\,dy\,dz$$

$$= \int_{z=1}^{5}\int_{\theta=0}^{2\pi}\int_{r=0}^{4} (2z-2)\, r\, dr\, d\theta\, dz$$

$$= \frac{r^2}{2}\Big|_0^4 \int_{z=1}^{5}\int_{\theta=0}^{2\pi} (2z-2)\, d\theta\, dz$$

$$I = 8 \cdot \theta\Big|_0^{2\pi} \int_{z=1}^{5} (2z-2)\,dz$$

$$= 16\pi \times 2 \left[\frac{z^2}{2} - z \Big|_1^5 \right]$$

$$= 32\pi \left[\frac{25}{2} - \frac{1}{2} - 5 + 1 \right]$$

$$= 32\pi \left[\frac{15}{2} + \frac{1}{2} \right] = 32 \times 8\pi$$

$$= 256\pi.$$

8(C) A particle moves with a central acceleration $\mu\left(\frac{3}{r^3}+\frac{d^2}{r^5}\right)$ being projected from a distance d at angle 45° with a velocity equal to that in a circle at the same distance. Prove that the time it takes to reach the centre of force is $\frac{d^2}{\sqrt{2\mu}}\left(2-\frac{\pi}{2}\right)$.

Sol^n: Here the central acceleration

$$P = \mu\left[\frac{3}{r^3}+\frac{a^2}{r^5}\right] \qquad \left[\because u = \frac{1}{r}\right]$$

If V is the velocity in a circle at a distance a under the same acceleration, then

$$\frac{V^2}{a} = [P]_{r=a} = \mu\left(\frac{3}{a^3}+\frac{a^2}{a^5}\right)$$

$$\Rightarrow V^2 = \frac{4\mu}{a^2} \text{ (or) } V = 2\sqrt{\mu}/a$$

The differential equation of path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(3u^3+a^2u^5) = \mu(3u+a^2u^3).$$

Multiplying bothsides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2+\frac{a^2}{2}u^4\right)+A \quad \text{---} (1)$$

where A is a constant.

But initially when $r=a$ i.e. $u=1/a$, $v=2\sqrt{\mu}/a$, $\phi=45^\circ$.

$p = r\sin\phi = a\sin\pi/4 = a/\sqrt{2}$ so that

$$\frac{1}{p^2} = u^2+(du/d\theta)^2 = 2/a^2$$

$\therefore$ from (1), we have $\frac{4\mu}{a^2} = h^2\cdot\frac{2}{a^2} = \mu\left(\frac{3}{a^2}+\frac{a^2}{2a^4}\right)+A$

$\therefore\ h^2 = 2\mu$ and $A = \mu/2a^2$

Substituting the values of h^2 and A in ①, we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2+\frac{a^2}{2}u^4\right)+\frac{\mu}{2a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2 = u^2+\frac{a^2}{2}u^4+\frac{1}{2a^2}$$

Putting $u = 1/r$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we have

$$\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2}+\frac{a^2}{2r^4}+\frac{1}{2a^2}$$

$$\Rightarrow 4a^2\left(\frac{dr}{d\theta}\right)^2 = 2a^2r^2+a^4+r^4 = (r^2+a^2)^2$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{r^2+a^2}{2a}$$ [-ve sign is taken because r decreases when θ increases].

we have $h = r^2\frac{d\theta}{dt} = r^2\frac{d\theta}{dr}\cdot\frac{dr}{dt}$

$$\Rightarrow \sqrt{2\mu} = -r^2\frac{2a}{(r^2+a^2)}\cdot\frac{dr}{dt}$$ [Substituting for h & $\frac{dr}{d\theta}$]

$$\Rightarrow dt = -\frac{2a}{\sqrt{(2\mu)}}\cdot\frac{r^2\,dr}{(r^2+a^2)}$$

Integrating between the limits $r=a$ to $r=0$, the required time t_1 from the distance a to the centre of force is given by

$$t_1 = -\frac{2a}{\sqrt{(2\mu)}}\int_{r=a}^{0}\frac{r^2\,dr}{r^2+a^2} = -\frac{2a}{\sqrt{(2\mu)}}\int_a^0\left(1-\frac{a^2}{r^2+a^2}\right)dr$$

$$= -\frac{2a}{\sqrt{(2\mu)}}\left[r-a\tan^{-1}\left(\frac{r}{a}\right)\right]_a^0$$

$$= -\frac{2a}{\sqrt{(2\mu)}}\left[\{0-a\tan^{-1}0\}-\{a-a\tan^{-1}(a/a)\}\right] = \frac{2a}{\sqrt{(2\mu)}}\left[a-a\tfrac{1}{4}\pi\right]$$

$$= \frac{a^2}{\sqrt{(2\mu)}}\left[2-\tfrac{1}{2}\pi\right]$$

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